Journal of Porous Media, 17 (8): 731–740 (2014)

RESISTANCE TO CREEPING FLOW AND PERMEABILITY OF STACKED SPHERES

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Original Manuscript Submitted: 8/22/2013; Final Draft Received: 2/20/2014

Six regularly packed beds (simple cubic, orthorhombic I and II, tetragonal sphenoidal, and rhombohedral I and II), treated as unit cells made of monosized spheres, were analyzed. A formula to calculate permeability of such beds at creeping flow conditions was proposed. It is based on similar assumptions as the Kozeny–Carman equation, but instead of mean porosity and tortuosity, their local values were taken into account. Two different values of the pore shape factors, regarding triangular and square pores, according to Boussinesq, were applied. The new formula, in its integral form, agrees better with experiments than those given by Slichter, Kozeny and Carman, Martin et al. as well as Franzen; it underestimates the analyzed packings' permeabilities by less than 22%, on average by 13%.

KEY WORDS: *seepage, periodic arrays of spheres, low Reynolds number*

1. INTRODUCTION

Fluid flows in beds and other porous media commonly occur in nature and are widely applied in engineering as well. They are the main topic of hydrogeology, petroleum engineering, etc. In environmental and chemical engineering, packed columns are used to perform separation processes, such as absorption, filtration, and stripping. The interaction between fluid and the porous medium is relatively well recognized in the simplest cases of a creeping flow in straight capillary tubes and in isotropic media. However, there are still many problems to be solved under more complex conditions using both analytical and numerical methods. One of them is a flow through a bed made of stacked spheres. It is known that spheres can be arranged in many regular packings. These arrangements were classified by Bravis as 14 lattices to determine crystal structures in three dimensions. They are also used as models of porous media made of spherical grains, thanks to their simple geometry and ease of its mathematical description. One important feature of the majority of these arrangements is their anisotropy, which influences vector-dependent properties. As will be shown, the existing semi-empirical and theoretical formulae predict permeability of six basic sphere arrays (simple cubic, orthorhombic I and II, tetragonal sphenoidal, and rhombohedral I and II; see Table 1), with unacceptable errors, except the one by Martin et al. (1951), which however is—at least partly—an empirical relationship only.

Such translationally invariant arrays of monosized spheres are attractive both from the practical and theoretical point of view. They may serve as special cases of more complex arrangements to check the validity of different mathematical models, particularly the correctness of their assumptions. Our goal was to elaborate a simplified mathematical model describing permeability and head losses during Newtonian fluid flow through regularly packed beds at low Reynolds numbers ($\mathrm{Re}_d = \rho U d/\mu < 1$).

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NOMENCLATURE

A	base cross-sectional area of unit cell or cross section of empty, very large tube, m^2	R_h	hydraulic radius, m
A_{0*}	ratio of the part of the tube cross-section area (unit cell) per one porous channel to the square $d \times d$ area	Re_d	$= \rho U d/\mu$ Reynolds number
A_p	pipe cross-sectional area or pores' area in a given cross section of unit cell, m^2	S_s	volumetric specific surface, m^2/m^3
b	number of layers which are cut by a plane passed through the bed perpendicularly to the main flow axis ($b = 1$ or 2)	T	$= L_e/L$ tortuosity
c_o	pore shape factor	U	$= q/A$ superficial velocity, i.e., average velocity of fluid based on the cross section of empty, very large tube, m s^{-1}
d	particle diameter, m	U_a	average interstitial velocity of fluid or mean velocity in an equivalent capillary tube, m s^{-1}
D	inner diameter of capillary tube, m	α	$= A_p/A$ surface porosity in a given cross section of unit cell
g	acceleration due to gravity, m s^{-2}	β	angle between direction of local path flow and the macroscopic flow direction, deg
H	head loss along a unit cell, m of fluid column	$\delta\kappa_r$	relative error of κ_r, %
Ha	$= d^2/\kappa$ Hagen number – relative hydraulic resistance	$\delta\kappa_{ra}$	mean absolute error of κ_r, %
k	pipe shape characteristic by Boussinesq	ε	voidage (volume porosity)
k_o	$= 1/c_o$	κ	permeability, m^2
L	depth of unit cell, m	κ_r	$= \kappa/d^2$ relative permeability
L_e	average effective path length of fluid in a unit cell, m	μ	dynamic viscosity of fluid, kg s^{-1} m^{-1}
q	flow intensity, m^3 s^{-1}	ν	kinematic viscosity of fluid, m^2 s^{-1}
		ρ	fluid density, kg m^{-3}
		χ_p	wetted perimeter of pipe or spheres in a given cross-section of unit cell, m

2. PREVIOUS STUDIES

Slichter (1899) developed a formula for velocity of fluid seeping through pores in regularly packed beds made of identical adjoining spheres. He determined a relationship between the volumetric porosity and the acute angle at the rhombic basis δ (Fig. 1), for a tetragonal sphenoidal ($\delta = \pi/3$) till a simple cubic packing ($\delta = \pi/2$) as well as a relation between the angle δ and minimum surface porosity, known also as a fractional free area (Martin et al., 1951) or an area porosity (Denys, 2003). The values of the studied porosities lay within the following ranges: $0.0931 \leq \alpha_{\min} \leq 0.2145$ and $0.2595 \leq \varepsilon \leq 0.4765$. Slichter assumed that an equivalent pore has a minimum cross section. The model of that type is called [according to Bear (1988)] *a capillary tube model*. The final formula for the dimensionless permeability reads

$$\frac{\kappa}{d^2} = \frac{\alpha_{\min}^2}{96\,(1-\varepsilon)} \tag{1}$$

where: κ – intrinsic permeability, d – particle diameter, $\alpha_{\min}$ – minimum surface porosity, ε – volume porosity.

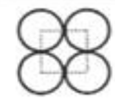
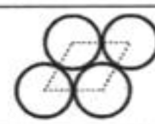
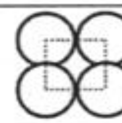
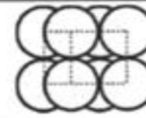
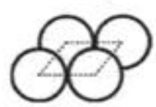
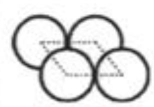
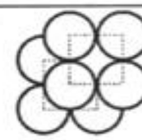
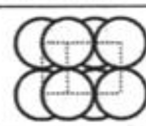
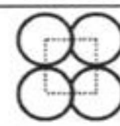
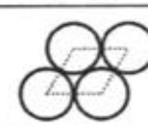
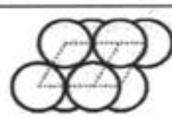
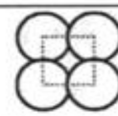
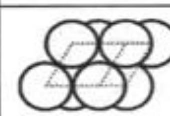
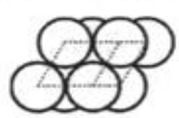
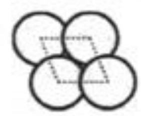
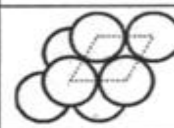
The advantage of Slichter's approach is there is no need for empirical coefficients.

Note that the volume porosity is just a surface porosity (fractional free area), averaged over bed depth L. Numerical values of the minimum surface porosity are close to the values of the mean surface porosity as well as the volume porosity ($\alpha_{\min} \approx \alpha_{av} = \varepsilon$) at a randomly dispersed (without any influence of the tube walls) arrangement of particles only.

Another capillary tube model, often used in porous media hydraulics, was developed by Kozeny (1927) on the base of an analytical solution of the Navier–Stokes equations, later on modified by Carman (1937). It allows estimation of the permeability using the following formula (Dullien 1979):

$$\kappa = \frac{c_o\,\varepsilon^3}{(1-\varepsilon)^2 S_s^2}\left(\frac{L}{L_e}\right)^2 \tag{2}$$

TABLE 1: Basic regular packing arrangements of spheres

Packing	Abbrev.	View		
		Front	Side	Top
Simple cubic	SC			
Orthorhombic II	OR II			
Body centered cubic	BCC			
Orthorhombic I	OR I			
Tetragonal sphenoidal	TS			
Rhombohedral hexagonal	RH I			

Note: Body centered cubic (BCC) packing is also known as rhombohedral II

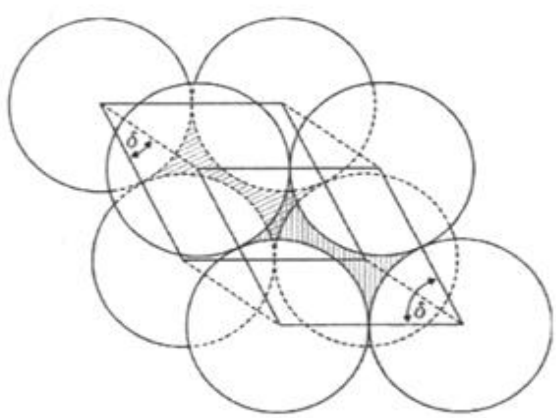


FIG. 1: Top view of spheres investigated by Slichter (1899)

where c_o – pore shape factor, ε – volume porosity and S_s – specific grain surface area per volume of solids, $L_e/L = T \geq 1$ – tortuosity.

Kozeny (1927) gave c_o values for different capillary tube cross sections: 0.5 (circle), 0.562 (square), 0.597 (equilateral triangle), and 0.667 (thin slot) using theoretical results obtained by Boussinesq (1868, 1914). Boussinesq derived them (fr. *caracteristiques de la forme*) for a laminar (potential) flow in prismatic pipes as

$$k = c_o \frac{A_p}{\chi_p^2} = c_o \frac{R_h}{\chi_p} \tag{3}$$

where A_p – pipe cross-section area, χ_p – wetted perimeter of the pipe, R_h – hydraulic radius, and obtained the values: $k = 0.0397$ (circle), 0.0351 (square), and 0.0288 (equilateral triangle).

For solid, randomly packed spherical grains Carman (1937) introduced into Eq. (2) $c_o(L/L_e)^2 = 0.2$, derived from empirical data, as well as $S_s = \pi d^2/(\pi d^3/6) = 6/d$ and obtained the well-known formula for the dimensionless permeability, valid for $\varepsilon < 0.5$:

$$\frac{\kappa}{d^2} = \frac{\varepsilon^3}{180\,(1-\varepsilon)^2} \tag{4}$$

It can be seen that formula (4) considers the volume porosity only. Taking the average tortuosity $T_a = L_e/L = \sqrt{2} - \pi/2 = 1.414–1.571$ as suggested by Carman for granular, randomly packed beds, one obtains $c_o = 0.2\,(L_e/L)^2 = 0.400–0.494$, which is close to the value adequate for a circular section of a representative pore (c_o = 0.5). However, the value accepted by Carman for "a narrow rectangular channel" $k_o = 1/c_o = 1/0.4 = 2.5$ was rather too high, as for a thin slot Kozeny (1927) suggested $k_o = 1/0.667 = 1.5$ instead of 2.5. In the well-known Ergun equation the constant 180 in the first right-hand term (based on the Blake–Kozeny equation for the creeping flow) is replaced by 150. Recently, Wu et al. (2008) have suggested that the numerical constant is proportional to the bed's tortuosity and should be taken as 72 T_a, which for granular, randomly packed beds may give values even lower than 150. The applicability of the last approach was confirmed by Schiavi et al. (2012). Another interpretation of the Kozeny–Carman constant—in terms of the fractal geometry—was proposed by Xu and Yu (2008) That concept is however not relevant to the seepage under saturated conditions in packed beds made of monosized spheres due to the lack of different scales of their structures It would be applied to solve some other problems, e.g. a percolation of fluid discharging from a point source and percolating through an array of regularly stacked spheres, which are beyond the scope of this paper.

Martin et al. (1951) showed that in the case of regularly packed beds made of spheres the flow direction is very important, as, e.g., the orthorhombic I packing with the same ε = 0.3954 as the orthorhombic II packing, has almost six times higher resistance to flow than the latter one. They investigated nine regular arrangements of spheres at Re_d = 0.5–10,000 and hypothesized that "In the case of stacked spheres ε may be thought of as the fractional free area of infinitesimal bed depth dz, as well as the fractional void volume over a finite height." A Newtonian fluid, seeping at superficial velocity U through a bed of depth L, made of stacked spheres of diameter d, creates the following pressure drop (Martin et al., 1951):

$$\Delta p = \frac{22\,L\,\mu}{d^2}\,U\,\frac{1}{L}\int_0^L \frac{b^4\,(1-\alpha)^2}{\alpha^3}\,dz \tag{5}$$

where b – number of layers which are cut by a plane passed through the bed perpendicularly to the main flow axis (b = 1 or 2).

It can be seen that in Eq. (5) volume porosity function $f(\varepsilon)$ from Eq. (4) has been substituted by a similar function of surface porosity $f(\alpha)$.

The hydraulic gradient can be then expressed in the following dimensionless form:

$$i = \frac{\Delta\,p}{\rho\,g\,L} = \frac{22\,\nu\,U}{g\,d^2}\,\frac{1}{L}\int_0^L \frac{b^4\,(1-\alpha)^2}{\alpha^3}\,dz \tag{6}$$

From Eq. (6) one can calculate the relative permeability as follows:

$$\frac{\kappa}{d^2} = \frac{L}{22}\left[\int_0^L \frac{b^4\,(1-\alpha)^2}{\alpha^3}\,dz\right]^{-1} \tag{7}$$

Franzen investigated fluid flows inside small channels imitating pores inside regularly packed beds of spherical beads (1977, 1979a) and the beds themselves (1979b) to determine the influence of the pore geometry on the pressure head loss. Six different types of packing were studied (Table 2); three of them were of cubic type and three of rhombic type in the middle of the unit cell depth [see Fig. 2(b)].

From data in Table 2 it can be stated that the orientation does not affect the volume porosity but it affects the surface porosity significantly. Franzen (1979b) investigated packed beds consisting of 14 layers of spheres of diameter d = 10 mm, within the Re_d = 0.5–1000. On the basis of the Hagen-Poiseuille equation

$$\Delta p = \frac{32\,\mu L_e U_a}{D^2} \tag{8}$$

where L_e – average effective path length equal to the length of an equivalent capillary tube, U_a – average velocity in the equivalent capillary tube of diameter D, with his own empirical correlation for the creeping flow ($\mathrm{Re}_d < 1$) conditions, Franzen (1979b) found

$$\frac{\kappa}{d^2} = \frac{\varepsilon^{1.5} A_{0*}^2}{143.2} \tag{9}$$

where A_{0*} – ratio of the part of the tube cross-section area (unit cell) per one porous channel to the square $d \times d$ area ($\sqrt{3}/4 = 0.433 \leq A_{0*} \leq 1.0$); for random packing $A_{0*} \approx$ 0.716.

TABLE 2: Basic characteristics of packings investigated by Franzen (1979b) and values of the Hagen numbers Ha

Type of packing	Volume porosity ε	Surface porosity α_{min} α_{max}	Mean tortuosity T_a	Relative area A_{0*}	Empirical values of Ha for $Re_d < 1$ at 20°C	Calculated values of $Ha = d^2/\kappa$ according to Eq. (9)	Error in Ha_F %
Simple cubic SC	0.4765	0.2145 1.0	1.0	1	420	436	+3.8
Orthorhombic II OR II	0.3954	0.2145 0.6355	4/3	1	600	576	−4.0
Rhombohedral II BCC	0.2595	0.2145 0.3494	$\sqrt{3}$	1.0 [a] 0.5 [b]	4400	1083 [a] 4333 [b]	75.4 −1.5
Orthorhombic I OR I	0.3954	0.0931 1.0	1.0	$\sqrt{3}/4$	3400	3072	−9.7
Tetragonal-sphenoidal TS	0.3019	0.0931 0.5791	4/3	$\sqrt{3}/4$	4300	4604	+4.6
Rhombohedral I (aba ...) RH I	0.2595	0.0931 0.4565	1.5	$\sqrt{3}/4$[a] $\sqrt{3}/2$[c]	5300	5777 [a] 1444 [c]	+9.0 72.8

[a] For one porous channel per d^2 at a section in the middle of a layer of spheres.
[b] For two (4 × 0.5) porous channels at sections laid $8^{-0.5}d$ above and below the middle of a unit cell [see Fig. 2(a)].
[c] For one porous channel at sections laid $6^{-0.5}$ d above and below the middle of a unit cell.

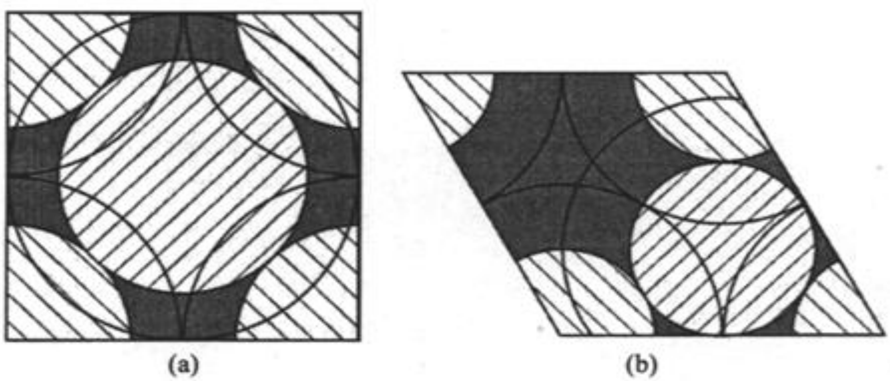


FIG. 2: Cross sections of rhombohedral packings: **(a)** BCC at section laid $8^{-0.5}$ d above and/or below the middle of the unit cell and **(b)** RH I at section in the middle of the unit cell depth

Franzen introduced a dimensionless similarity number Ha (Hagen number) which is a reciprocal of the dimensionless permeability, i.e., $Ha = d^2/\kappa$, which can be interpreted as a dimensionless resistivity. A comparison of the calculated and empirical values of the dimensionless resistivity is shown in Table 2. It can be seen that the values of Ha correspond well to each other (relative error $< \pm 10\%$), provided that for the rhombohedral II (body-centered cubic, BCC) arrangement $A_{0*} = 0.5$ instead of 1.0 is used, typical value for cubic packings and pores in the shape of diamond (◇) or four-pointed star. Franzen noticed that at sections laid $8^{-0.5}$ $d = 0.354$ d above and below the middle of a layer there are two (4 × 0.5) porous channels of diamond-shaped cross section in a unit cell [Fig. 2(a)]. Franzen's observation that in the rest of the packings the number of enclosed pore cross sections is identical to the number of pore openings in the upper sphere layer is not correct, because in the case of RH I, there are two openings (enclosed pores) in the upper sphere layer and only one enclosed pore (denoted in gray color) in the middle of the cell [Fig. 2(b)]; therefore, one should take doubled $A_{0\bullet} = 2\sqrt{3}/4 = \sqrt{3}/2 = 0.866$ $A_{0\bullet} = (\sqrt{3}/4)/2 = \sqrt{3}/8 = 0.217$ instead of $\sqrt{3}/4 = 0.433$, which leads to a relatively large error in Ha = d^2/κ equal to −72.8% (Table 2) and even much greater in κ/d^2, equal to 266% (Table 3).

Larson and Higdon (1989) analyzed creeping fluid flows through three cubic arrays [simple cubic (SC), BCC, and face-centered cubic (FCC)] of solid spheres by using a numerical collocation method based on a set of expansion functions for the solution of Stokes flow in terms of Lamb's equations in spherical coordinates. Their

TABLE 3: Values of permeability and their relative errors related to the values estimated from Franzen's (1979b) experiments.

Authors of experimental data *Sphere diameter*		**Dimensionless permeability κ_r of packings of spheres**					
		SC	OR II	BCC	OR I	TS	RH I
Martin et al. (1951) *d = 0.79 cm*		$4.0 \cdot 10^{-3}$	$2.94 \cdot 10^{-3}$	$3.02 \cdot 10^{-4}$	$4.39 \cdot 10^{-4}$	$3.83 \cdot 10^{-4}$	$3.02 \cdot 10^{-4}$
d = 1.59 cm		$9.88 \cdot 10^{-4}$	$7.25 \cdot 10^{-4}$	$7.46 \cdot 10^{-5}$	$1.08 \cdot 10^{-4}$	$9.46 \cdot 10^{-5}$	$7.46 \cdot 10^{-5}$
recalculated for d = 1.0 cm		$2.50 \cdot 10^{-3}$	$1.83 \cdot 10^{-3}$	$1.89 \cdot 10^{-4}$	$2.74 \cdot 10^{-4}$	$2.39 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$
Franzen (1979b) d = 1.0 cm		$2.38 \cdot 10^{-3}$	$1.67 \cdot 10^{-3}$	$2.27 \cdot 10^{-4}$	$2.94 \cdot 10^{-4}$	$2.36 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$
Authors of equation	Eq.	Dimensionless permeability κ_r of packings of spheres of d = 1.0 cm *Relative error $\delta\kappa_r$, %* *Mean absolute error $\delta\kappa_{ra} = \frac{1}{6}\sum_{i=1}^{6} \lvert\delta\kappa_{ri}\rvert$*					
Slichter	(1)	$9.16 \cdot 10^{-4}$ *−61.5*	–	–	–	$1.29 \cdot 10^{-4}$ *−45.3*	–
		53.4% (for SC and TS only)					
Kozeny–Carman	(4)	$2.19 \cdot 10^{-3}$ *−8.0*	$9.40 \cdot 10^{-4}$ *−43.7*	$1.77 \cdot 10^{-4}$ *−22.0*	$9.40 \cdot 10^{-4}$ *219.7*	$3.14 \cdot 10^{-4}$ *33.1*	$1.77 \cdot 10^{-4}$ *−6.3*
		55.5%					
Martin et al.	(7)	$2.74 \cdot 10^{-3}$ *13.1*	$2.20 \cdot 10^{-3}$ *31.7*	$1.72 \cdot 10^{-4}$ *−24.2*	$2.54 \cdot 10^{-4}$ *−13.6*	$2.16 \cdot 10^{-4}$ *−8.5*	$1.91 \cdot 10^{-4}$ *1.1*
		15.4%					
Franzen	(9)	$2.30 \cdot 10^{-3}$ *−3.4*	$1.74 \cdot 10^{-3}$ *4.2*	$2.28 \cdot 10^{-4}$ *0.4*	$3.26 \cdot 10^{-4}$ *10.9*	$2.17 \cdot 10^{-4}$ *−8.1*	$6.92 \cdot 10^{-4}$ *266.1*
		48.9%					
This study	(13)	$2.00 \cdot 10^{-3}$ *−16.0*	$1.54 \cdot 10^{-3}$ *−7.8*	$2.22 \cdot 10^{-4}$ *−2.2*	$2.32 \cdot 10^{-4}$ *−21.1*	$1.96 \cdot 10^{-4}$ *−16.9*	$1.63 \cdot 10^{-4}$ *−13.8*
		13.0%					
Stokes-Lamb equations solved by Larson and Higdon (1989)		$2.53 \cdot 10^{-3}$ *6.3*	–	$2.94 \cdot 10^{-4}$ *29.5*	–	–	–
		17.9% (for SC and BCC only)					

results concerning permeability of SC and BCC packings were compared with empirical data and results of our calculations in Table 3.

3. DERIVATION OF NEW PERMEABILITY FORMULA

We consider a creeping flow (seepage under saturated conditions) of a Newtonian fluid through unit cells of six regular packings (Table 1) along the vertical z axis (normal to each cell's top horizontal plane). Our basic assumption is that the particle Reynolds number is much less than 1. A unit cell is independent of the neighboring ones. An inspiration for our approach presented below was formula (5) with its enigmatic factor b. Its twice squared value ($b^4 = 1$ or 16) has suggested that by cutting two layers of spheres one gets a significantly greater wetted perimeter and correspondingly smaller local hydraulic radius.

Applying the Hagen–Poiseuille law (8), substituting D for $4R_h$, and introducing a local tortuosity T as a function of z, a head loss along an infinitesimal depth of the bed Δz can be expressed as

$$\Delta H = \frac{\nu U_a T}{c_o g R_h^2}\Delta z = \frac{\nu\, U\, T^2}{c_o\, \alpha\, g\, R_h^2}\Delta\, z \qquad (10)$$

where $U_a = U \cdot T/\alpha$ – average fluid velocity in an equivalent pore of a given cross-section, $R_h = A_p(z)/\chi_p(z) = \alpha(z)A/\chi_p(z)$ – hydraulic radius of the equivalent pore (Niven 2002); the rest of notations are as under Eq. (3).

Integration of Eq. (10) over the bed depth L gives the head loss in the form

$$\int_0^H dH = H = \frac{\nu\, U}{c_o\, g}\int_0^L \frac{T^2}{\alpha\, R_h^2}\, dz \qquad (11)$$

Dividing it by L and rearranging Eq. (11) we get the mean hydraulic gradient

$$\frac{H}{L} = i = \frac{\nu U}{c_o\, g}\frac{1}{L}\int_0^L \frac{T^2}{\alpha R_h^2}\, dz \qquad (12)$$

and the corresponding dimensionless permeability:

$$\frac{\kappa}{d^2} = \kappa_r = \frac{c_o L}{d^2}\left[\int_0^L \frac{T^2}{\alpha R_h^2}\, dz\right]^{-1} \qquad (13)$$

The integrals in Eqs. (7) and (13) were calculated numerically using trapezoidal rule.

For the sake of simplicity the values of pore shape coefficients were assumed to be constant and equal to $c_o = 0.562$ for cubic packings (as for square pores) and $c_o = 0.597$ for rhombic packings [as for triangular pores – see comment to Eq. (3)]. Tortuosity (Fig. 3) and surface porosity (Fig. 4) were estimated using Franzen's (1979b) sketches and geometrical relationships, respectively. In each of these sketches a central line, normal to the enclosed pores at inlet to and outlet from a unit cell, as well as to the enclosed pores inside the unit cell, was drawn in direction of the main stream (Fig. 5). Mean tortuosity factor T_a was then estimated by summation of the flow path lengths L_e and subsequent division by the depth of the unit cell L. For example: for BCC and tetragonal sphenoidal (TS) $T_a = (\sqrt{3}/6 + \sqrt{3}/3 + \sqrt{3}/6)/(\sqrt{3}/2) = 1.333$. A local tortuosity T is interpreted here as the reciprocal of cosine of the angle β (Fig. 5) between the direction of a given central line and the z axis. This type of tortuosity can be called, after Sobieski et al. (2012), an overall

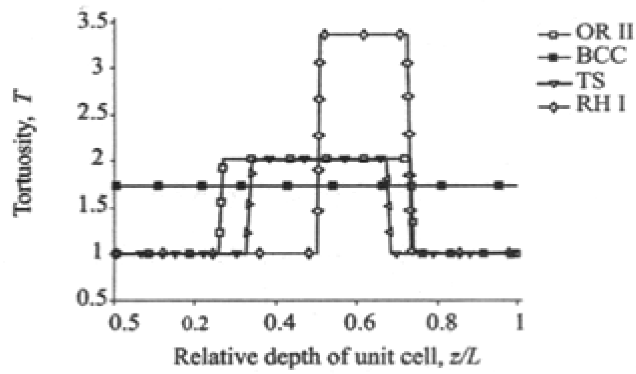


FIG. 3: Tortuosity of four investigated packings. For SC and OR I: $T = 1.0$

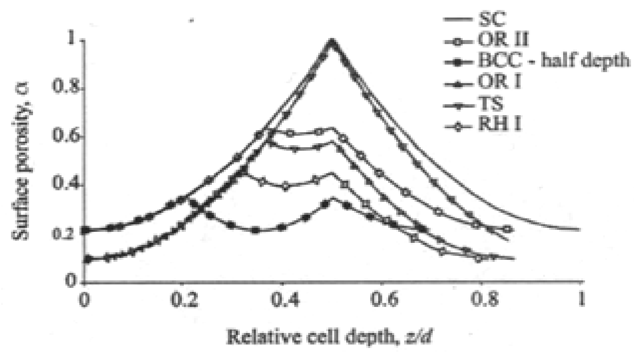


FIG. 4: Surface porosities of investigated packings

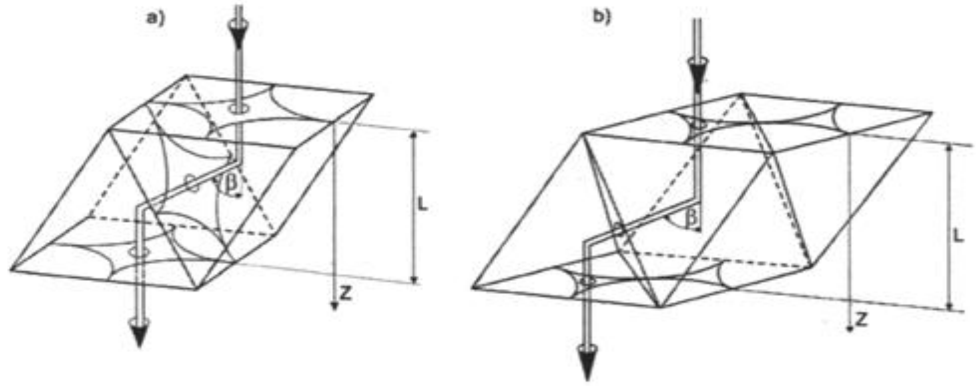


FIG. 5: Sketches of central lines within unit cells of OR II **(a)** and RH I **(b)** packings (adapted from Franzen, 1979b)

(equivalent) hydraulic tortuosity as it represents a relative length of one equivalent pore passing through a unit cell. In fact, in some packings there are locally three (in RH I) or four (in BCC) identical flow paths around a single sphere, fortunately with the same tortuosity and the same hydraulic radii (provided that the pores are identical) as the single equivalent hydraulic radius. Taking into account all streamlines, the total tortuosity would be greater than the general one but there are also some dead zones within pores which are usually ignored, although they increase the diffusivity and micro-tortuosity.

4. DISCUSSION

To validate the above presented formulae one has to compare calculated values with the corresponding measured ones. Such data were provided by several researches; the most detailed—as far as we know—by Martin et al. (1951) and Franzen (1979b). The latter data seem to be more reliable due to better documentation of the experimental conditions. Unfortunately, both sources of empirical data did not provide estimation of uncertainties.

In Table 3 dimensionless values of hydraulic permeability and their relative errors ($\delta\kappa_r = 100\ (\kappa_{rc}-\kappa_{re})/\kappa_{re}$), where the lower index c denotes calculated value and e – experimental data by Franzen (1979b) are shown. It can be seen that Slichter's approach underestimates the permeability of regularly packed beds (SC and TS) due to the assumption that the equivalent pore is that of the minimum cross section. The Kozeny–Carman's equation and the corresponding dimensionless permeability described by Eq. (4) are not sensitive to anisotropy (orientation), therefore they give unacceptably high errors for the OR I packing; e.g., $\delta\kappa_r = -220\%$. Replacement of the numerical constant 180 by 72 T_a, suggested by Wu et al. (2008) has improved the results in the cases of OR II and BCC packings only. Applying the Yu and Li formula Eq. (4) in the paper by Wu et al. (2008) for the tortuosity as a function of porosity, one may obtain an additional improved result (for RH 1 packing), but still on average out of all six packings not better than those delivered by the original Kozeny–Carman formula (4). The highest discrepancy between calculated and empirical data ($\delta\kappa_r = -369\%$) showed once again the OR I packing. Much better agreement was obtained using the semi-empirical Eq. (7) by Martin et al. (1951). Franzen's formula [Eq. (9)] is good for all packings except from rhombohedral I (RH I) with the stipulation discussed in Section 2. Our results slightly underestimate the packings' permeabilities by less than 22%, on average – 13%. They have occurred on average better than those delivered by the rest of the analyzed formulae. Even numerical calculations done by Larson and Higdon (1989) gave a relatively high discrepancy (relative error $\delta\kappa_r = 29.5\%$) in the case of BCC packing. In the case of SC packing the agreement was much better ($\delta\kappa_r = 6.3\%$), as good as that obtained by smooth particle hydrodynamic simulations (Holmes et al., 2011).

Our results confirm also the correctness of Carman's (1937) modification of the Blake–Kozeny equation (Kozeny 1927) consisting in the introduction of T^2 instead of T.

The most probable sources of errors in our model are simplifications of complex shapes of pores and their tortuosity. Our simplified procedure of numerical integration assumes tortuosity patterns shown in Fig. 3 as discrete (piecewise) instead of gradual ones. A smoothing procedure, as suggested by Sobieski et al. (2012), would improve the obtained results. Application of the mean tortuosity T_a instead of the local ones $T(z)$ gave worse results in the cases when the latter was not constant over the cell depth, i.e., for OR II, TS, and RH I packings. The question: "how to estimate a representative value of tortuosity?" is still open due to complexity of pore geometry (Duda et al., 2011), even in regular sphere packings. It is known that in unit cells, apart from the main path, there are also some other, less significant paths of fluid flow with different tortuosities. To estimate the hydraulic resistance of regular sphere packings, one has to take into account the local tortuosity rather than any value averaged over a unit cell volume.

5. CONCLUSIONS

Hydraulic resistance to creeping flow and permeability of beds made of regularly stacked spheres depend strongly on *local* values of their surface porosity and tortuosity; therefore, the models based on these parameters averaged over the bed's bulk volume may often give significant errors.

Slichter's approach overestimates head losses at slow fluid flow through regularly packed beds (SC and TS) and underestimates their permeabilities due to the assumption that the equivalent pore is that of the minimum cross section.

Franzen's formula (9) is in fact an empirical one and ambiguous in interpreting the numbers of openings as well as enclosed pores in planes normal to the superficial flow velocity.

A new method to calculate the head loss and permeability of packed beds made of stacked spheres, substituting the volume porosity by the local surface porosities as well as the mean tortuosity by its local equivalents, is proposed. Acceptable agreement with experimental data by Martin et al. (1951) and Franzen (1979b) confirms the correctness of this approach.

ACKNOWLEDGMENT

We gratefully acknowledge the discussions on the topic of tortuosity with Dr. Maciej Matyka of Wrocław University (Poland) and his help in acquisition of the literature and Grzegorz Jaszczak for drawing figures.

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